Malaysian Journal of Mathematical Sciences 14(S) December : 155–169 (2020)
Special Issue: 2nd International Conference on Applied & Industrial Mathematics and Statistics 2019 (ICoAIMS 2019)

MALAYSIAN JOURNAL OF MATHEMATICAL SCIENCES

Journal homepage: http://einspem.upm.edu.my/journal

Numerical Solution of Fractional Pantograph Differential Equation via Fractional Taylor Series Collocation Method

Bilal, M. [1,2], Rosli, N. * [1], Mohd Jamil, N. [1], and Ahmad, I. [3]

[1] *Center for Mathematical Sciences, College of Computing and Applied Sciences, Universiti Malaysia Pahang, Pahang*
[2] *Department of Mathematics, University of Lahore, Gujrat Campus, Pakistan*
[3] *Department of Mathematics, University of Gujrat, Hafiz Hayat Campus, Pakistan*

E-mail: norhayati@ump.edu.my
* *Corresponding author*

Received: 15 January 2020
Accepted: 20 November 2020

ABSTRACT

In this paper, a collocation method which based on polynomial approximation of Taylor's series is proposed to approximate the solution of fractional pantograph differential equations (FPDE). The collocation method with truncated Taylor's polynomial is shown to be an applicable technique in solving FDDE. Some examples of the non-linear fractional pantograph differential equations are solved and compared with the exact solution to confirm the accuracy and applicability of the collocation method with Taylor's polynomial.

Keywords: Collocation Method, Taylor Polynomials, Fractional Pantograph Differential Equations.

1. Introduction

Fractional calculus is a calculus of derivative and integral which are widely used to study the behaviour of real phenomena in science and engineering (see Wang (2013)). It becomes important in recent years as the fractional calculus can explain the complex system with non-linear behaviour and long term memory. Fractional pantograph differential equation (FPDE) is a class of functional differential equations with a proportional delay which is capable of modeling the systems that subject to the memory or after effects. It has gained popularity in various areas of science and engineering, namely in material and mechanics (see Agarwal et al. (2010)), dynamics of viscoelastic materials (see Benchohra et al. (2008)), wave propagation (Butzer and Westphal (2000)), systems identification, electromagnetism (Gorenflo et al. (2002)), visco-elastic materials (Koeller (1984)), signal processing, continuum and statistical (Lakshmikantham (2008)), spherical flames (Saeedi et al. (2013)), fluid mechanics (Rabiei and Ordokhani (2019)) and anomalous diffusion (Loh et al. (2018)).

Fractional models are more consistent with the real phenomena than the integer models (Doha et al. (2014)). It is due to the fact that fractional derivatives and integrals enable to describe the memory and hereditary properties inherent (Doha et al. (2014)). Due to its complexity of the delay argument and fractional form, the analytical solution of FPDEs is hard to be found. Hence there is a growing interest in researching numerical methods for solving FPDEs. Amongst of the cited works by Isah and Phang (2018) who proposed an operational matrix of derivative with Genocchi polynomials, Heris and Javidi (2017) who presented fractional backward differential formulas with periodic and antiperiodic conditions and Xu and Lin (2016) who considered a simplified reproducing kernel method.

In this paper, a Taylor collocation method is proposed for solving FPDEs. The solutions are obtained in terms of fractional order Taylor's series. A matrix representation of the collocation method of fractional pantograph differential equation via Taylor's polynomials are derived. The results obtained indicate good performance compare to the existing methods in the literature. The generalized FPDEs is given by

$$D^{i\alpha}u(x) = \sum_{r=1}^{m} P_r(x)u(q_r x - c) + g(x) \qquad 0 \leq x \leq T \tag{1}$$

subject to initial condition

$$\sum_{n=0}^{m-1} a_{ni}u^n(c) = \zeta_i \tag{2}$$

where a_{ni}, q_r, c are the real or complex coefficients, while $P_r(x)$ and $g(x)$ are given continuous function in the interval [0,T]. A collocation method of FPDE with the following Taylor polynomial

$$u_N(x) = \sum_{i=0}^{N} \frac{(x-c)^{i\alpha}}{\Gamma(i\alpha+1)}(D^{i\alpha})(c) \tag{3}$$

is introduced.

The remaining part of the paper is organized as follows: Section 2 defines the fractional derivative of Riemann Liouvile and Caputo. Both derivatives are important approaches to generalize the notion of differentiation to fractional orders. The fundamental relation of the derivatives is carried out in Section 3. A method of the solution is presented in Section 4. In Section 5, examples of FPDEs are solved and compared with the reported works that have been done by previous researchers. Conclusion remarks are provided in Section 6.

2. Preliminaries

2.1 Fractional Derivative

The fundamental definitions and the properties of fractional calculus that will help us to calculate the fractional derivative are presented in this section. There are many fundamental definitions in literature for fractional derivatives (see Kilbas et al. (2006), Wang (2013)). One of them which is important is Riemann Liouville's approach. Although it provides the basis of the development of the theory in fractional calculus, it is difficult to be applied when dealing with initial value problem. To handle such problems, Caputo's definition which is a modification of Riemann–Liouville definition was introduced. We first give the definition of Riemann-Liouville integral, in which the fractional integral operator I of a function $g(x)$ is defined as follows.

2.2 Definition 1

The Riemann Liouville integral I of fractional order α of $g(x)$ is given by

$$I^\alpha g(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} g(x) d\tau \qquad x > 0, \alpha \in \mathbb{R}^+ \tag{4}$$

where $\Gamma(.)$ is a Gamma function. The fractional derivative of order $\alpha > 0$ due to Riemann Liouville is defined by

$$(D_I^{\alpha} g)(x) = \left(\frac{d}{dx}\right)^m (I^{m-\alpha} g)(x) \qquad (\alpha > 0, m-1 < \alpha < m). \tag{5}$$

Two basic properties of Riemann Liouville's fractional integral I^{α} are:

$$I^{\alpha} T^{\beta} g(x) = I^{\alpha+\beta} g(x), \alpha > 0, \beta > 0$$
$$I^{\alpha} T^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} x^{\alpha+\beta}.$$

2.3 Definition 2

The fractional derivative D^{α} of $g(x)$ in Caputo's sense

$$D^{\alpha} g(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-\tau)^{n-\alpha-1} g^{(n)}(\tau) d\tau \qquad n-1 < \alpha \leq n, n \in \mathbb{N} \tag{6}$$

The properties of Caputo fractional are:

$D^{\alpha} C = 0, \quad C$ is constant

$D^{\alpha} x^{\beta} = 0, \quad \beta \geq \lceil \alpha \rceil$

$D_c^{\alpha} (x-c)^{\beta} \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} (x-c)^{\beta-\alpha}, \qquad \beta \in \mathbb{N} \bigcup \{0\}, \beta \geq \lceil \alpha \rceil \text{or} \beta \in \mathbb{N}, \beta > \lfloor \alpha \rfloor$

where $\lceil \alpha \rceil$ is the smallest numbers greater or equal than α and $\lfloor \alpha \rfloor$ is the largest numbers less or equal than α.

3. Fundamental Matrix Relation

In this section, we propose a fundamental matrix relation of the solution $u(x)$ in (1) subject to the initial condition (2) defined by the truncated of Taylor's polynomial (3). In matrix form, $u(x)$ in equation (1) defined by a truncated Taylor's series (3) can be written as

$$u(x) = \mathbf{X M_0 A} \tag{7}$$

where X is a matrix function which depend on x and defined as

$$\mathbf{X} = [1 \quad (x-c)^{\alpha} \quad (x-c)^{2\alpha} \quad (x-c)^{3\alpha} \cdots (x-c)^{N\alpha}]$$

$$\mathbf{M_0} = \begin{pmatrix} \frac{1}{\Gamma(1)} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{\Gamma(\alpha+1)} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{\Gamma(2\alpha+1)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \frac{1}{\Gamma(N\alpha+1)} \end{pmatrix}; \mathbf{A} = \begin{pmatrix} D^{0\alpha}u(c) \\ D^{1\alpha}u(c) \\ D^{2\alpha}u(c) \\ D^{3\alpha}u(c) \\ \vdots \\ D^{N-1\alpha}u(c) \\ D^{N\alpha}u(c) \end{pmatrix}.$$

The matrix representation of the function $D^{\alpha}u(x)$ will become

$$D^{\alpha}u(x) = \mathbf{X}\mathbf{M_0}\mathbf{A}$$

The function $D^{\alpha}\mathbf{X}$, can be computed as

$$D^{\alpha}\mathbf{X} = [1 \quad D^{\alpha}(x-c)^{\alpha} \quad D^{\alpha}(x-c)^{2\alpha} \quad D^{\alpha}(x-c)^{3\alpha} \cdots D^{\alpha}(x-c)^{N\alpha}]$$

$$= [0 \quad \tfrac{\Gamma(\alpha+1)}{\Gamma(1)} \quad \tfrac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}(x-c)^{\alpha} \quad \tfrac{\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)}(x-c)^{2\alpha} \cdots \tfrac{\Gamma(N\alpha+1)}{\Gamma((N-1)\alpha+1)}(x-c)^{N\alpha}]$$

$$= \mathbf{X}\mathbf{M_1}$$

where

$$\mathbf{M_1} = \begin{pmatrix} 0 & \frac{\Gamma(\alpha+1)}{\Gamma(1)} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \frac{\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{\Gamma(N\alpha+1)}{\Gamma((N-1)\alpha+1)} \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

and

$$\mathbf{X} = [1 \quad (x-c)^{\alpha} \quad (x-c)^{2\alpha} \cdots (x-c)^{(N-1)\alpha}].$$

By the same way, the matrix representation of $D^{2\alpha}u(x)$ can be obtained as

$$D^{2\alpha}u(x) = \mathbf{X}\mathbf{M}_2\mathbf{M}_0\mathbf{A}. \tag{8}$$

Similarly, for any i^{th}, it can be written as

$$D^{i\alpha}u(x) = \mathbf{X}\mathbf{M_i}\mathbf{M_0}\mathbf{A} \tag{9}$$

where

$$\mathbf{M_i} = \begin{pmatrix} 0 & 0 & \cdots & \frac{\Gamma(\alpha+1)}{\Gamma(1)} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & \frac{\Gamma(N-i+1\alpha+1)}{\Gamma((N-i)\alpha+1)} \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

4. Method of Solution

The fundamental matrix equation corresponding to equation (1) is presented in this section. Let define collocation points as follows

$$x_i = \frac{i}{N} \qquad i = 0, 1, 2, 3, \cdots, n-1. \tag{10}$$

By using the collocation points, the system of matrix equations (1) is transformed as

$$\big(\mathbf{X}\mathbf{M}_j\mathbf{M}_0 - \sum_{r=1}^{m-1} \mathbf{P}_r\mathbf{X}\mathbf{B}_{q,c}x_i\mathbf{M}_j\mathbf{M}_0\big)\mathbf{A} = \mathbf{G}(x_i) \tag{11}$$

where

$$\mathbf{W} = \mathbf{X}\mathbf{M}_i\mathbf{M}_0 - \sum_{r=0}^{m} \mathbf{P}_r\mathbf{X}\mathbf{B}_{q,c}(x_i)\mathbf{M}_0 \tag{12}$$

$$\mathbf{X} = \begin{pmatrix} 1 & (x_0-c)^{\alpha} & (x_0-c)^{2\alpha} & \cdots & (x_0-c)^{N\alpha} \\ 1 & (x_1-c)^{\alpha} & (x_1-c)^{2\alpha} & \cdots & (x_1-c)^{N\alpha} \\ 1 & (x_2-c)^{\alpha} & (x_2-c)^{2\alpha} & \cdots & (x_2-c)^{N\alpha} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & (x_N-c)^{\alpha} & (x_N-c)^{2\alpha} & \cdots & (x_N-c)^{N\alpha} \end{pmatrix},$$

$$\mathbf{P} = \begin{pmatrix} p_r(x_0) & 0 & 0 & \cdots & 0 \\ 0 & p_r(x_1) & 0 & \cdots & 0 \\ 0 & 0 & p_r(x_2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & p_r(x_N) \end{pmatrix},$$

$$\mathbf{B_{q,c}} = \begin{pmatrix} (q_r x_0 - c)^{0\alpha} & 0 & 0 & \cdots & 0 \\ 0 & (q_r x_1 - c)^{1\alpha} & 0 & \cdots & 0 \\ 0 & 0 & (q_r x_2 - c)^{2\alpha} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (q_r x_N - c)^{N\alpha} \end{pmatrix}$$

and

$$\mathbf{G} = \begin{pmatrix} g(x_0) \\ g(x_1) \\ g(x_2) \\ g(x_3) \\ \vdots \\ g(x_{N-1}) \\ g(x_N) \end{pmatrix}.$$

Hence, equation (13) corresponding to equation (1) can be written in the form of

$$\mathbf{ZA} = \mathbf{G} \qquad \text{or} \qquad [\boldsymbol{Z}; \boldsymbol{G}]; \qquad \boldsymbol{Z} = [\boldsymbol{z_{i,j}}] \quad \boldsymbol{i, j = 0, 1, 2, \cdots, N} \tag{13}$$

$$\mathbf{Z} = \mathbf{TM}_i \mathbf{M}_0 - \sum_{r=0}^{m} \mathbf{P}_r \mathbf{TB}_{q,c} \mathbf{M}_0. \tag{14}$$

We produce the representation of the equation (2) in matrix $i = 0, 1, \cdots, m-1$

$$Y_i = \mathbf{X(c)M_k M_0} = [\mathbf{y_{i0}} \quad \mathbf{y_{i1}} \quad \mathbf{y_{i2}} \quad \cdots \mathbf{y_{iN}} \quad] = [\zeta_\mathbf{i}]. \tag{15}$$

The unknown values of the fractional Taylor coefficients

$$D_*^{k\alpha} u(c), k = 0, 1, \cdots, N$$

related with the approximate solution of the problem (1) with initial condition (2) can be found by replacing the m^{th} row matrix in $[Y_i; \zeta_i]$ by the first m row of the matrix in (12). Hence, the augmented matrix is

$$[\bar{\mathbf{Z}};\bar{\mathbf{G}}] = \begin{bmatrix} z_{00} & z_{01} & z_{02} & \cdots & z_{0N} & ; & g(t_0) \\ z_{10} & z_{11} & z_{12} & \cdots & z_{1N} & ; & g(t_1) \\ z_{20} & z_{21} & z_{22} & \cdots & z_{2N} & ; & g(t_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & ; & \vdots \\ z_{(N-m)0} & z_{(N-m)1} & z_{(N-m)2} & \cdots & z_{(N-m)N} & ; & g(t_{(N-m)}) \\ y_{00} & y_{01} & y_{02} & \cdots & y_{0N} & ; & \zeta_0 \\ y_{10} & y_{11} & y_{12} & \cdots & y_{1N} & ; & \zeta_1 \\ y_{20} & y_{21} & y_{22} & \cdots & y_{2N} & ; & \zeta_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & ; & \vdots \\ y_{(m-1)0} & y_{(m-1)1} & y_{(m-1)2} & \cdots & y_{(m-1)N} & ; & \zeta_{m-1} \end{bmatrix}. \tag{16}$$

In another form of matrix equation we have

$$\bar{\mathbf{Z}}\mathbf{A} = \bar{\mathbf{G}}. \tag{17}$$

If $det\ \bar{\mathbf{Z}} \neq 0$, then $\bar{\mathbf{Z}}$ is an invertible matrix and we can write equation (1) as

$$\mathbf{A} = (\bar{\mathbf{Z}})^{-\mathbf{1}}\bar{\mathbf{G}}. \tag{18}$$

The matrix $\mathbf{A}$ is uniquely determined and the solution of (1) is determined by truncated Taylor series

$$u_N(x) = \sum_{i=0}^{m} \frac{(x-c)^{k\alpha}}{\Gamma(k\alpha+1)}(D^{k\alpha}y)(c).$$

4.1 Residual Error

The present section considers the residual error for problem (1) with initial condition (2). The residual error is a way to measure the efficiencies of the corresponding numerical method for the case where the exact result is not known. The error function can be defined as follows

$$e_N(x) = u(x) - u_N(x) \tag{19}$$

where $u(x)$ and $u_N(x)$ are the exact and approximate solution of (1), respectively. Substituting $u_N(x)$ into (1) leads to

$$\left(D^{i\alpha}u_N(x) - \sum_{r=0}^{m} p_r(x)u_N(q,x)\right) = g(x) + \vartheta_N(x) \tag{20}$$

where $\vartheta_N(x)$ is the perturbation term that obtained by substituting the computed solution $u_N(x)$ into equation (19), i.e.

$$\vartheta_N(x) = \Big(D^{i\alpha}u_N(x) - \sum_{r=0}^{m} p_r(x)u_N(q,x)\Big) - g(x). \tag{21}$$

By subtracting equation (21) from (1) and using (19), the error function $e_N(x)$ satisfies

$$-\vartheta_N(x) = \Big(D^{i\alpha}e_N(x) - \sum_{r=0}^{m} p_r(x)e_N(q,x)\Big). \tag{22}$$

5. Illustrative Examples

Several numerical examples are presented in this section to illustrate the effectiveness of the collocation method with TCM for solving the FPDE. The algorithm to simulate the approximate results are computed in MATLAB R2017b with double precision and the residual analysis is carried out in Minitab 17.

Example 1
Consider the FPDE in Sherif et al. (2014)

$$D^{\alpha}u(x) = -u(x) + u\left(\frac{x}{2}\right) + \frac{3x^2}{4} + \frac{2x^{2-\alpha}}{\Gamma(3-\alpha)} \tag{23}$$

with $u(0) = 0$ and the exact solution is $u(x) = x^2$ for $\alpha = 1$. The exact solution is unavailable for $\alpha \neq 1$ and we need to compute the residual error to measure the efficiency of the method. The numerical results of Example 1 are summarised in Table 1. The comparison is made with the existing results that were reported in Sherif et al. (2014). It can be seen that the simulated results obtained by using the collocation method with Taylor's polynomial produce low values of the error. This indicates that the proposed method has better efficiency compared to the reported method in Sherif et al. (2014).

Table 1: Result and Error of Example 1 with comparison with Spline Function in Sherif et al. (2014)

α	x	Exact Solution	Result by Spline Function	Error by Spline Function	TCM Result	TCM Error
$\alpha = 0.1$	0.01	0.0001	0	3.36E-08	0.0001	1.61E-10
	0.02	0.0004	7.55819E-05	1.13E-06	0.0004	9.46E-11
	0.03	0.0009	0.000265263	1.75E-06	0.0009	6.98E-11
	0.04	0.0016	0.000557029	2.13E-06	0.0016	5.64E-11
	0.05	0.0025	0.000945615	2.33E-06	0.0025	4.78E-11
$\alpha = 0.2$	0.01	0.0001	0	3.24E-08	0.0001	6.41E-09
	0.02	0.0004	0.000807984	1.16E-06	0.0004	3.97E-09
	0.03	0.0009	0.000273582	1.61E-06	0.0009	3.02E-09
	0.04	0.0016	0.000560054	1.74E-06	0.0016	2.50E-09
	0.05	0.0025	0.000932151	1.62E-06	0.0025	2.16E-09
$\alpha = 0.3$	0.01	0.0001	0	4.20E-08	0.0001	3.70E-11
	0.02	0.0004	8.96535E-05	1.19E-06	0.0004	2.42E-11
	0.03	0.0009	0.000287934	1.42E-06	0.0009	1.90E-11
	0.04	0.0016	0.000570342	1.24E-06	0.0016	1.60E-11
	0.05	0.0025	0.000926655	7.28E-07	0.0025	1.41E-11
$\alpha = 0.4$	0.01	0.0001	0	6.66E-08	0.0001	4.92E-12
	0.02	0.0004	0.000101184	1.18E-06	0.0004	3.37E-12
	0.03	0.0009	0.000305454	1.13E-06	0.0009	2.72E-12
	0.04	0.0016	0.000583095	5.98E-07	0.0016	2.35E-12
	0.05	0.0025	0.000922591	2.61E-07	0.0025	2.10E-12

Example 2

Consider the fractional pantograph differential equation in Rahimkhani et al. (2017)

$$D^{\alpha}u(x) = -\frac{5}{6}u(x) = 4u\left(\frac{x}{2}\right) + 9u\left(\frac{x}{3}\right) + x^2 - 1 \qquad (24)$$

subject to initial condition $u(0) = 1$ and the exact solution is $u(x) = 1 + \frac{67x}{6} + \frac{1675x^2}{72} + \frac{12157x^3}{1296}$ when $\alpha = 1$.

Table 2 illustrates the simulated results of Example 2. For α=1, TCM shows low values of the error, hence indicate better efficiency of the method. When $\alpha \neq 1$, the exact solution of the equation is not known, thus require to compute the residual as presented in Section 4. The residual error is computed for $N = 75, 45, 25, 9$ and the error obtained is used as a reference solution. With that reference solution we exclude the absolute error (AE) when the $\alpha = 0.95, 0.75, 0.50$.

Table 2: Results and Error for Example 2 with different values of α.

x	Results					Errors			
		α=1	α=0.95	α=0.75	α=0.50	α=1	α=0.95	α=0.75	α=0.50
	Exact	N=75	N=45	N=25	N=9	N=75	N=45	N=25	N=9
0	1.000000	1.000000	1.000000	1.000000	1.000000	2.00E-15	6.15497E-10	1.3592E-10	1.9984E-15
0.1	2.358686	2.358686	2.636999	4.951775	25.77266673	1.42E-14	6.92957E-08	4.2414E-08	4.4180E-07
0.2	4.238932	4.238932	4.875551	10.40047	66.10073739	2.93E-14	1.28794E-07	8.9202E-08	1.1330E-06
0.3	6.697021	6.697021	7.811405	17.72148	123.43423580	4.35E-14	2.0716E-07	1.5212E-07	2.1150E-06
0.4	9.789235	9.789235	11.51221	27.06577	198.58307480	6.22E-14	3.06175E-07	2.3243E-07	3.4030E-06
0.5	13.57186	13.57186	16.04379	38.57752	292.42240570	9.06E-14	4.26636E-07	3.3142E-07	5.0100E-06
0.6	18.10117	18.10117	21.4715	52.39899	405.84164530	1.21E-13	5.70542E-07	4.5018E-07	6.9530E-06
0.7	23.43345	23.43345	27.86045	68.67102	539.72632030	1.60E-13	7.39831E-07	5.8995E-07	9.2470E-06
0.8	29.62499	29.62499	35.27559	87.53317	694.95137300	2.06E-13	9.37001E-07	7.5184E-07	1.1910E-05
0.9	36.73206	36.73206	43.78173	109.1238	872.37884520	2.34E-13	1.16506E-06	9.3718E-07	1.4950E-05
1	44.81096	44.81096	53.44354	133.5800	1072.85737300	7.11E-14	1.26051E-06	1.1489E-06	1.8380E-05

Example 3

Consider the FPDE in Rahimkhani et al. (2017)

$$D^{\alpha}u(x) = -u(x)+0.1u\left(\frac{4x}{5}\right)+0.5D^{\alpha}u\left(\frac{4x}{5}\right)+(0.32x-0.5)e^{-0.8x}+e^{-x} \quad (25)$$

for $0 \leq \alpha \leq 1$. The exact solution is given by $u(x) = xe^{-x}$. The simulated results are illustrated in Table 3. Table 3 represents the results and the error of Example 3. When α=1, TCM for FPDE improve the error, where α do not have an exact solution, for those value of α we calculate the residual error. For that problem, we calculate the residual error at a different number of $N = 75, 45, 25, 9$, where we have a better residual error, we use it as a reference solution.

Table 3: Results and Error for Example 3 with different values of α.

x	Results					Errors			
		α=1	α=0.95	α=0.75	α=0.50	α=1	α=0.95	α=0.75	α=0.50
	Exact	$N = 75$	$N = 75$	$N = 45$	$N = 75$	$N = 75$	$N = 75$	$N = 45$	$N = 75$
0	0	0	0	0	0	0	0	0	0
0.1	0.090484	0.090484	0.101910	0.155837	0.229542	2.78E-17	-3.60957E-07	6.33692E-07	3.07125E-06
0.2	0.163746	0.163746	0.176398	0.225374	0.271652	2.22E-16	-2.79096E-07	3.81196E-07	1.26318E-06
0.3	0.222245	0.222245	0.232910	0.267522	0.289870	1.39E-16	-2.38176E-07	2.58127E-07	7.44440E-07
0.4	0.268128	0.268128	0.275439	0.293870	0.297502	3.89E-16	-2.09831E-07	1.84823E-07	5.08182E-07
0.5	0.303265	0.303265	0.306764	0.309895	0.299293	5.00E-16	-1.91855E-07	1.36711E-07	3.76132E-07
0.6	0.329287	0.329287	0.329009	0.318765	0.297508	4.44E-16	-1.87386E-07	1.03364E-07	2.93082E-07
0.7	0.347610	0.347610	0.343864	0.322503	0.293405	3.89E-16	-1.89026E-07	7.95523E-08	2.36749E-07
0.8	0.359463	0.359463	0.352702	0.322488	0.287753	6.66E-16	-1.94398E-07	6.13491E-08	1.96391E-07
0.9	0.365913	0.365913	0.356654	0.319697	0.281052	2.78E-16	-1.87638E-07	4.44758E-08	1.66314E-07
1.0	0.367879	0.367879	0.356652	0.314844	0.273644	4.44E-15	-1.20788E-06	-1.02230E-09	1.4327E-07

In Example 1, the exact solution for the FPDE is unavailable when $\alpha = 0.1, 0.2, 0.3, 0.4$. However, FPDE solutions were not available in the literature for computing errors to reflect the stability and accuracy of the TCM. We used cubic polynomial to compute the residuals and mean square error (MSE) of the FPDE at different values of alpha and matrix sizes. The estimated value computed through polynomials which provided least MSE, statistical residuals (observed value-estimated value), the highest coefficient of determination (R^2) and most precise prediction interval (PI) were considered as reference or baseline values for comparison in example 2 and 3 separately. Based on these baseline values, errors were computed and reported in tables 2 and 3. The example of the polynomial with the polynomial equation, (R^2) and PI was given in Figure 1.

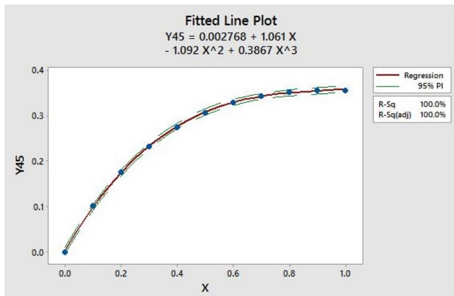

Figure 1: Baseline polynomial for computation of error in Example 2

Example 4

Consider the FPDE in

$$D^{\alpha} = 1 - 2u^2(\frac{x}{2}), \quad 0 < \alpha \leq 1, 0 < x \leq 1 \tag{26}$$

$u(x) = \sin(x), \quad -1 \leq x \leq 0.$

The exact solution, when $\alpha = 1$ is $u(x) = \sin(x)$

Table 4: Results and Error for Example 4 with different values of α.

	$\alpha = 1$		$\alpha = 0.55$	
x	Exact	CWM	TCM	TCM
0	0	1.23E-22	3.22E-14	5.21E-13
0.125	0.123674733	1.71E-12	2.43E-14	7.65E-13
0.250	0.247403959	2.47E-12	1.62E-14	2.95E-13
0.375	0.366272529	9.36E-12	8.01E-14	5.90E-13
0.500	0.479425538	1.79E-11	4.17E-13	3.82E-11
0.625	0.585097272	1.89E-11	6.28E-13	2.85E-11
0.750	0.68163876	3.04E-12	5.27E-13	2.19E-11
0.875	0.767543502	1.64E-12	3.29E-13	3.21E-11
1.000	0.841470984	4.51E-10	1.11E-13	1.51E-10

The exact solution is given by $u(x) = \sin(x)$. The simulated results are illustrated in Table 4. Table 4 represents the results and the error of Example 4, when α=1, TCM for FPDE improve the error as we compare with the reference result in $\alpha \neq 1$ do not have an exact solution, for those value of α we calculate the residual error. For that problem, we calculate the residual error at a different number of $N = 0.55$, where we have a better residual error, we use it as a reference solution.

6. Concluding Remarks

In this work, a collocation method based on the truncating of Taylor's polynomial is presented to solve FPDE. It can be concluded that TCM performs well as indicated by low values of error obtained in three illustrative examples. The statistical technique to measure the residual error is applied when the exact solutions are not available. Taylor's polynomial is more preferable to embeded in the collocation method since it is easy to program.

Acknowledgement

We would like to thank Research & Innovation Department, Universiti Malaysia Pahang (UMP) for financial support through the Universiti Malaysia Pahang grants RDU1703190 and RDU180301.

References

Agarwal, R. P., Zhou, Y., and He, Y. (2010). Existence of fractional neutral functional differential equations. *Computers & Mathematics with Applications*, 59(3):1095–1100. `https://doi.org/10.1016/j.camwa.2009.05.010`.

Benchohra, M., Henderson, J., Ntouyas, S., and Ouahab, A. (2008). Existence results for fractional order functional differential equations with infinite delay. *Journal of Mathematical Analysis and Applications*, 338(2):1340–1350. `https://doi.org/10.1016/j.jmaa.2007.06.021`.

Butzer, P. L. and Westphal, U. (2000). An introduction to fractional calculus. In *Applications of Fractional Calculus in Physics*, pages 1–85. World Scientific. `https://doi.org/10.1142/9789812817747_0001`.

Doha, E., Bhrawy, A., Baleanu, D., and Hafez, R. (2014). A new jacobi rational–gauss collocation method for numerical solution of generalized pantograph equations. *Applied Numerical Mathematics*, 77:43–54.

Gorenflo, R., Mainardi, F., Moretti, D., and Paradisi, P. (2002). Time fractional diffusion: a discrete random walk approach. *Nonlinear Dynamics*, 29(1-4):129–143. https://doi.org/10.1142/9789812817747_0001.

Heris, M. S. and Javidi, M. (2017). On fractional backward differential formulas for fractional delay differential equations with periodic and anti-periodic conditions. *Applied Numerical Mathematics*, 118:203–220.

Isah, A. and Phang, C. (2018). Operational matrix based on genocchi polynomials for solution of delay differential equations. *Ain Shams Engineering Journal*, 9(4):2123–2128.

Kilbas, A. A., Srivastava, H. M., and Trujillo, J. J. (2006). Preface. In *Theory and Applications of Fractional Differential Equations*, pages vii–x. Elsevier.

Koeller, R. (1984). Applications of fractional calculus to the theory of viscoelasticity. *Journal of Applied Mechanics*, 51(2):299–307. https://doi.org/10.1115/1.3167616.

Lakshmikantham, V. (2008). Theory of fractional functional differential equations. *Nonlinear Analysis: Theory, Methods & Applications*, 69(10):3337–3343. https://doi.org/10.1016/j.na.2007.09.025.

Loh, J. R., Isah, A., Phang, C., and Toh, Y. T. (2018). On the new properties of caputo–fabrizio operator and its application in deriving shifted legendre operational matrix. *Applied Numerical Mathematics*, 132:138–153.

Rabiei, K. and Ordokhani, Y. (2019). Solving fractional pantograph delay differential equations via fractional-order boubaker polynomials. *Engineering with Computers*, 35:1431–1441. https://doi.org/10.1007/s00366-018-0673-8.

Rahimkhani, P., Ordokhani, Y., and Babolian, E. (2017). Numerical solution of fractional pantograph differential equations by using generalized fractional-order bernoulli wavelet. *Journal of Computational and Applied Mathematics*, 309:493–510. https://doi.org/10.1016%2Fj.cam.2016.06.005.

Saeedi, L., Tari, A., and Masuleh, S. H. M. (2013). Numerical solution of some nonlinear volterra integral equations of the first kind. *Applications & Applied Mathematics*, 8(1):65–77. https://doi.org/10.14317/jami.2013.065.

Sherif, M., Abouelfarag, I., and Amer, T. (2014). Numerical solution of fractional delay differential equations using spline functions. *International Journal of Pure and Apllied Mathematics*, 90(1):73–83. 10.12732/ijpam.v90i1.10.

Wang, Z. (2013). A numerical method for delayed fractional-order differential equations. *Journal of Applied Mathematics*, 2013:1–7. https://doi.org/10.1155/2013/256071.

Xu, M.-Q. and Lin, Y.-Z. (2016). Simplified reproducing kernel method for fractional differential equations with delay. *Applied Mathematics Letters*, 52:156–161.